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Jungck Modified SP-Iterative Scheme

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Abstract

In this paper, we introduce Jungck-modified SP-iteration and we study its rate convergence with respect to the rate convergence Jungck-Mann, Jungck-Ishikawa, Jungck-Noor, Jungck-SP, Jungck-CR, Jungck-Karahan, Jungck-Picard S- iterative schemes.

Keywords: Jungck iterations; convergence; rate of convergence.

1. Introduction and Preliminaries

In 1976 Jungck has studied the common fixed points of self-mappings [1], but he has noted that Pfeffer [2] had discussed anointer dependence between the commuting mapping and the fixed point concepts, thus he highlights this interdependence but in a more general context.

1.1 ***Proposition [1]:*** Let S be a mapping on a set X into itself. Then Shas a fixed point if and only if there is a constant mapping $T: X \to X$which commutes with S i.e $T(S(x)) = S(T(x))$ for all x in X.

Hence he has extended this proposition to a common fixed point concept.

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1.2 Theorem [1]: Let S be a continuous mapping of a complete matric space $(X.d)$ intoitself. Then S has a fixed point in X if and only if there exists Lin (0.1) and a mapping $T: X \rightarrow X$ which commutes with S andsatisfies

$T(X) \subset S(X)$ and $d\big(T(x).T(y)\big) \leq kd\big(S(x).S(y)\big)$ (1.1)

For all $x.y \in X$. Indeed S and T have common fixed point if (1.1) holds.

From the proof of the above theorem, the Jungck Picard iterative scheme is appeared

1.3 Definition [1]: Let C be a subset of a set $X.\ T.S: C \rightarrow C$ be two self mappings such that $T(C) \subseteq S(C)$then for $x_0 \in C$

$S(x_{n+1}) = T(x_n)\ ;\ n \in \mathbb{N}.$ (1.2)

Also Jungck contraction mapping is introduced in [1].

1.4 Definition [1]: let C be a nonempty subset of a normed space X, then$T.S: C \rightarrow C$ are called Jungck-contraction mapping if there exist $k \in (0.1)$such that

$\|Tx - Ty\| \leq k\|Sx - Sy\|$ for all $x.y \in C$

Jungck- non expansive mapping if we have

$\|Tx - Ty\| \leq \|Sx - Sy\|$ for all $x.y \in C$

After that Jungck and his colleagues tried to weak the commuting condition. Thus in 1996 Jungck and his colleagues [3] introduced the following definition that dependence on confidence point concepts, recall that a point $x \in C$ is acoincidence point of S and T if $Sx = Tx$[4], [5].

1.5 Definition[6]: Let $(E.d)$ be a metric space, C be a nonempty subset of E, and S and T be self-maps of C.The pair $(S.T)$ is calledweakly compatible if $STx = TSx$ for all $x \in C(S.T)$.

Since 2005 many authors introduce different types of Jungck scheme like as the following.

1.6 Definition: Let C be a subset of a set$X, T.S: C \rightarrow C$ be two self mappings such that $T(C) \subseteq S(C)$, $\{\alpha_n\}.\{\beta_n\}.\{\gamma_n\}$ be real sequencesin [0.1], then

1 **The Jungck-Mann scheme**: define the sequence $\{S\hat{t}_n\}$ by Singh et al [7] as:

$$\begin{cases} \hat{t}_0 \in C \\ S\hat{t}_{n+1} = (1-\alpha_n)S\hat{t}_n + \alpha_n T\hat{t}_n \end{cases} \quad (1.3)$$

2 **The Jungck- Ishikawa scheme**: define the sequence$\{Sa_{n+1}\}$, by Olatinwo[8], as

$$\begin{cases} a_0 \in C \\ Sa_{n+1} = (1-\alpha_n)Sa_n + \alpha_n Tb_n \\ Sb_n = (1-\beta_n)Sa_n + \beta_n Ta_n \,;\, n \in \mathbb{N} \end{cases} \tag{1.4}$$

3 **The Jungck-Noor scheme**: define the sequence $\{Sw_n\}$, by [9] as

$$\begin{cases} w_0 \in C \\ Sw_{n+1} = (1-\alpha_n)Sw_n + \alpha_n Tu_n \\ Su_n = (1-\beta_n)Sw_n + \beta_n Tv_n \\ Sv_n = (1-\gamma_n)Sw_n + \gamma_n Tw_n \,;\, n \in \mathbb{N} \end{cases} \tag{1.5}$$

4 **The Jungck-SP scheme**: define the sequence $S\hat{a}_n$, by Chugh and Kumar [10] as

$$\begin{cases} \hat{a}_0 \in C \\ S\hat{a}_{n+1} = (1-\alpha_n)S\hat{b}_n + \alpha_n T\hat{b}_n \\ S\hat{b}_n = (1-\beta_n)S\hat{c}_n + \beta_n T\hat{c}_n \\ S\hat{c}_n = (1-\gamma_n)S\hat{a}_n + \gamma_n T\hat{a}_n \,;\, n \in \mathbb{N} \end{cases} \tag{1.6}$$

5 **The Jungck-CR scheme**: define the sequence $\{S\widehat{w}_n\}$[10] as

$$\begin{cases} \widehat{w}_0 \in C \\ S\widehat{w}_{n+1} = (1-\alpha_n)S\hat{u}_n + \alpha_n T\hat{u}_n \\ S\hat{u}_n = (1-\beta_n)S\widehat{w}_n + \beta_n T\hat{v}_n \\ S\hat{v}_n = (1-\gamma_n)S\widehat{w}_n + \gamma_n T\widehat{w}_n \,;\, n \in \mathbb{N} \end{cases} \tag{1.7}$$

Motivated by the above definition, we introduce the following Jungck type iterative procedures:

1.7 Definition: Let C be a subset of a set X , $T.S: C \to C$ be two self mappings such that $T(C) \subseteq S(C)$,$\{\alpha_n\}.\{\beta_n\}.\{\gamma_n\}$be real sequences in [0.1]

1 **The Jungck-Karahan scheme**: define the sequence $\{S\widetilde{w}_n\}$ as

$$\begin{cases} \widetilde{w}_0 \in C \\ S\widetilde{w}_{n+1} = (1-\alpha_n)T\widetilde{w}_n + \alpha_n T\tilde{u}_n \\ S\hat{u}_n = (1-\beta_n)S\widetilde{w}_n + \beta_n T\tilde{v}_n \\ Sv_n = (1-\gamma_n)S\widetilde{w}_n + \gamma_n T\widetilde{w}_n \,;\, n \in \mathbb{N} \end{cases} \tag{1.8}$$

2 **The Jungck-Picard-S scheme**: define the sequence $\{S\hat{x}_n\}$ as

$$\begin{cases} \hat{x}_0 \in C \\ S\hat{x}_n = T\hat{y}_n \\ S\hat{y}_n = (1-\alpha_n)S\hat{x}_n + \alpha_n T\hat{z}_n \\ S\hat{z}_n = (1-\beta_n)S\hat{x}_n + T\beta_n\hat{x}_n \,;\, n \in \mathbb{N} \end{cases} \tag{1.9}$$

3 **The Jungck-modified SP-iteration scheme**: define the sequence $\{Sx_n\}$as

$$\begin{cases} x_0 \in C \\ Sx_n = Ty_n \\ Sy_n = (1-\alpha_n)Sz_n + \alpha_n Tz_n \\ Sz_n = (1-\beta_n)Sx_n + \beta_n Tx_n \, ; n \in \mathbb{N} \end{cases} \tag{1.10}$$

Where $\{\alpha_n\}.\{\beta_n\}.\{\gamma_n\}$be real sequences in [0.1].

In 2013, Aggarwal and his colleagues [11] introduce the concept of Jungcknon-expansive mapping

1.8 Defention1.4 [11]: let C be a nonempty subset of normed linear space$X$, then$T.S: C \to C$ are called Jungck-nonexpansive mapping if we have

$\|Tx - Ty\| \leq \|Sx - Sy\|$ for all $x.y \in C$

1.9 Lemma [12]: Let X be uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequence in X such that $\lim_{n\to\infty} \sup x_n \leq a$, $\lim_{n\to\infty} \sup y_n \leq a$ and $\lim_{n\to\infty} \sup\|\mathrm{t_n x_n} + (1-\mathrm{t_n})\mathrm{y_n}\| = a$, holds for some $a \geq 0$, then $\lim_{n\to\infty}\|\mathrm{x_n} - \mathrm{y_n}\| = 0$.

In the next three sections we establish our main results. In the second section, we will discuss the results of strong convergence and common fixed point of Jungck-modified SP-iteration. In the third section, compare speed of Jungck-modified SP-iteration with various Jungck-iterative shames with certain condition. In the fourth section, we prove the stability result of and established data dependence result of Jungck-modified SP-iteration.

2. Strong Convergence and Common Fixed Point

In this section we will discuss the strong convergence for the iterative (1.10).The following proportion states the images of the sequence $\{x_n\}$ under $T.S$ are closed to each other as far as possible.

2.1 Proposition: Let C be a nonempty closed convex subset of a Banach space X, $T.S: C \to C$be self-mappings satisfying Jungck- nonexpansive mappings. If $\{Sx_n\}$ generated by (1.10).with for some λ ; $0 \leq \lambda \leq \beta_n.\alpha_n \leq 1$ for all $n \in \mathbb{N}$, then:

1 $\lim_{n\to\infty}\|Sx_n - P\|$ exists for all $p \in F$

2 $\lim_{n\to\infty}\|Tx_n - Sx_n\| = 0$

Proof:part(1): Let p be a common fixed point of S and$T$, then:

$$\begin{aligned} \|Sx_{n+1} - P\| = \|Ty_n - P\| \leq \|Sy_n - P\| &= \|(1-\alpha_n)Sz_n + \alpha_n Tz_n - P\| \\ &\leq (1-\alpha_n)\|Sz_n - P\| + \alpha_n\|Tz_n - P\| \\ &\leq (1-\alpha_n)\|Sz_n - P\| + \alpha_n\|Sz_n - P\| = \|Sz_n - P\| \end{aligned} \tag{2.1}$$

Now,

$\|Sz_n - P\| = \|(1-\beta_n)Sx_n + \beta_n Tx_n - p\| \leq (1-\beta_n)\|Sx_n - P\| + \beta_n\|Tx_n - p\| = \|Sx_n - p\|$ (2.2)

From (2.1) and (2.2) we have, $\|Sx_{n+1} - P\| \leq \|Sx_n - p\|$ for all $n \in \mathbb{N}$. This shows that $\|Sx_n - p\|$ is non-increasing, thus the prove of part(1) is complete.

Part(2): Now we will prove that $\lim_{n\to\infty}\|Sx_n - P\| = 0$:

By part(1) there exists $c \in \mathbb{R}$; $\lim_{n\to\infty}\|Sx_n - P\| = c$ implies that

$\lim_{n\to\infty} sup\|Sx_n - P\| \leq c$ (2.3)

Since $\|Tx_n - P\| \leq \|Sx_n - p\|$ implies that

$\lim_{n\to\infty} sup\|Tx_n - p\| \leq c$ (2.4)

From (2.1) we get, $c \leq \lim_{n\to\infty} inf\ \|Sz_n - p\|$

By (2.2) we get:

$\lim_{n\to\infty} sup\ \|Sz_n - p\| \leq c$ (2.5)

Form (2.4) (2.5)we obtain that:

$\lim_{n\to\infty} sup\ \|Sz_n - p\| \leq c \leq \lim_{n\to\infty} inf\ \|Sz_n - p\|$ (2.6)

So $\lim_{n\to\infty}\|Sz_n - p\| = c$. Hence, this implies that

$c = \lim_{n\to\infty}\|Sz_n - p\| = \lim_{n\to\infty}\|(1-\beta_n)(Sx_n - p) + \beta_n(Tx_n - p))\|$ (2.7)

By using lemma (1.3) and from (2.3), (2.4),(2.7) we get $\lim_{n\to\infty}\|Tx_n - Sx_n\| = 0$.

The following corollary show that $\{Sx_n\}$defined by the three-stepiterative Jungck-modified SP-iterative (1.10) converge strongly toa fixed point of $F$with a certain conditions.

2.2 Corollary: Suppose that all hypotheses of proposition (2.1) are satisfied. If $\lim_{n\to\infty} inf\ d(Sx_n\ .F) = 0$ or $\lim_{n\to\infty} sup\ d(Sx_n\ .F) = 0$, then $\{Sx_n\}$ converge strongly to a fixed point of $S$and T.

Proof: By Proposition (2.1l), we know $\lim_{n\to\infty}\|Sx_n - P\|$ exists for all $p \in F$, therefore $d(Sx_n\ .F)$ exists for all $p \in F$. But by hypothesis we get $\lim_{n\to\infty} inf\ d(Sx_n\ .F) = \lim_{n\to\infty} sup\ d(Sx_n\ .F) = 0$. Hence$d(Sx_n\ .F) = 0$.We will show that $\{Sx\}_n$ is a Cauchy sequence in C, let $\varepsilon > 0$. Since $d(Sx_n\ .F) = 0$, thenthere exists $n_0 \in \mathbb{N}$ such that $d(Sx_n\ .F) < \frac{\varepsilon}{4}$ for all $n \geq n_0$ .In particular, $\inf\{\|Sx_{n_0} - p\|\} \leq \frac{\varepsilon}{4}$. Hence there exists $p^* \in F$ such that $\|Sx_{n_0} - p^*\| \leq \frac{\varepsilon}{2}$.Now for all $m.n \leq n_0$,$\|Sx_{n+m} - Sx_n\| \leq \|Sx_{n+m} - p^*\| + \|Sx_n - p^*\| \leq \varepsilon$. Hence $\{Sx_n\}$

is Cauchy sequence, but C is closed in a Banach space,therefore it must be converge to a point z in C. Since $\lim_{n\to\infty} d(Sx_n . F) = 0$, gives that $z \in F$, thus $\{Sx_n\}$ Converge strongly to a fixed point $z \in F$.■

The following corollary prove that $\{Sx_n\}$ defined by the three-stepiterative Jungck-modified SP-iterative (1.10) converges strongly to afixed point in F with different certain conditions.

2.3 Corollary: Suppose all hypothesis of (2.1) are satisfied. If there exists a non-decreasing function $f: [0.\infty) \to [0.\infty)$ with $f(0) = 0$ and $f(r) > r$ for all $r \in (0.\infty)$ such that $f(d(Sx.F)) \le \|Sx - Tx\|$ for all $x \in C$, then $\{Sx_n\}$ converge strongly to a point of F.

Proof: From Proposition(2.1), $\lim_{n\to\infty}\|Tx_n - Sx_n\| = 0$ .thus, $\lim_{n\to\infty} f\big(d(Sx.F)\big) \le \lim_{n\to\infty}\|Sx - Tx\| = 0$, that is, $\lim_{n\to\infty} f\big(d(Sx.F)\big) = 0$. Since $f: [0.\infty) \to [0.\infty)$ is non-decreasingfunction satisfying $f(0) = 0. f(r) > r$ for all $r \in (0.\infty)$, hence $\lim_{n\to\infty} d(Sx_n . F) = 0$ .Thus the conditions of Corollary(2.2) are hold, therefore $\{Sx_n\}$ converges strongly to a point of F.■

The following theorem show that Jungck-modified SP-iterative (1.10) convergence strongly to unique common fixed point of F.

2.4 Theorem: Let X be a Banach space, and $S.T: X \to X$ are Jungck-contraction mappings, weakly compatible assume that $T(X) \subseteq S(X)$, let $Sq = Tq = p$, and $\{Sx_n\}$ generated by (1.10) with some λ ; $0 \le \lambda \le \beta_n . \alpha_n \le 1$. Then the Jungck modified SP-iteration converge strongly to p . Moreover p is the unique common fixed point of (S,T).

Proof: Since $S.T$ are Jungck contraction mappings then

$$\begin{aligned}\|Sx_{n+1} - p\| &= \|Ty_n - p\| \\ &\le L\|Sy_n - p\| \\ &= L[1 - \alpha_n(1 - L)]\|Sz_n - p\| \\ &= L[1 - \lambda(1 - L)]\|Sz_n - p\| \quad (2.8)\end{aligned}$$

But,

$$\begin{aligned}\|Sz_n - p\| &= \|(1 - \beta_n)Sx_n + \beta_n Tx_n - p\| \\ &\le (1 - \beta_n)\|Sx_n - p\| + \beta_n\|Tx_n - p\| \\ &\le (1 - \beta_n(1 - L))\ \|Sx_n - p\| \\ &\le (1 - \lambda(1 - L))\ \|Sx_n - p\| \qquad (2.9)\end{aligned}$$

Substituting (2.8) in (2.9) we have, $\|Sx_{n+1} - p\| \le \big[L\big(1 - \lambda(1 - L)\big)\big]^2 \|Sx_0 - p\|$. By indication we get

$$\|Sx_{n+1} - p\| \le [\,L\big(1 - \lambda(1 - L)\big)]^{2n}\ \|Sx_0 - p\|$$

Take the limit for both sides and since $(1-\lambda(1-L)) \in (0.1)$, thus $\{Sx_n\}$ converges strongly to p.

the uniqueness of the fixed point p of $(S.T)$ comes from $S.T$ are Jungck-contraction mappings and $0 < k < 1$.

Now since $S.T$ are weakly compatibleand $p = Tq = Sq$ then $Tp = TSq = STq$. Hence $Tp = Sp$ and $TTp = TSp = STp$, therefore Tp is coincidence of $S.T$ but the coincidence point is unique, so$p = Tp$. Thus $Sp = Tp = p$. Therefore $p$is the unique common fixed point of $(S.T)$.■

3. Rate of Converge

We turning our iteration to the speed of varies Jungck iterative schemes with Jungck-modified SP-iterative scheme for a pair of Jungck contraction mappings using certain condition. First we give a theorem that say Jungck-modified SP-iteration (1.10) converges faster than Jungck-Mann iteration (1.3}), Jungck-Ishikawa iterative (1.4) to $p \in F$.

3.1 Theorem: Let X be a normed space and Cbe a nonempty closed convex subset of $X, S.T: C \to C$ be self mappings satisfied Jungck-contractionmappings, such that $T(C) \subseteq S(C)$. If there exist λ ; $0 \le \lambda \le \beta_n . \alpha_n \le 1$ for all n, then theJungck-modified SP-iteration (1.10) convergefaster than Jungck-Mann(1.3) and Jungck-Ishikawa (1.4) to $p \in F$.

Proof: Let $p \in F$, for Jungck modified SP-iteration (1.10), we obtain

$$\begin{aligned} \|Sx_{n+1} - p\| &= \|Ty_n - p\| \le L\|Sy_n - p\| \\ &\le L[1-\alpha_n(1-L)]\|(1-\beta_n)Sx_n + \beta_n Tx_n - p\| \\ &\le L[1-\alpha_n(1-L)]\big[\big(1-\beta_n(1-L)\big)\big]\|Sx_n - p\| \\ &\le \big(1-\lambda(1-L)\big)^2\|Sx_n - p\| \end{aligned}$$

By induction, we get, $\|Sx_{n+1} - p\| \le [\,L(1-\lambda\,(1-L)\,)^2]^n\,\|Sx_0 - p\|$

Put:

$$JMS_n \coloneqq [\,L\big(1-\lambda(1-L)\big)]^{2n}\ \|Sx_0 - p\| \qquad (3.1)$$

From Jungck-Mann iteration (1.3), and by induction, we obtain that

$$\|S\hat{t}_{n+1} - p\| \le [(1-\lambda(1-L)]^n\|S\hat{t}_0 - p\|$$

Put,

$$JMA_n \coloneqq [(1-\lambda(1-L)]^n\|S\hat{t}_0 - p\|$$

From Jungck-Ishikawa process (1.4), we obtain that

$$\|Sa_{n+1} - p\| \leq [1 - \lambda(1 - L) - L\lambda^2(1 - L)]^n \|Sa_0 - p\|$$

Put,

$$JIS_n := [1 - \lambda(1 - L) - L\lambda^2(1 - L)]^n \|Sa_0 - p\|$$

Now after simple compute we get:

$$\frac{JMS_n}{JMA_n} = \frac{[\,k(1-\lambda(1-k))]^{2n} \quad \|Sx_0 - p\|}{[1-\lambda(1-L)]^n \qquad \|S\hat{t}_0 - p\|} \text{as} \quad n \to \infty$$

and

$$\frac{JMS_n}{JIA_n} = \frac{[\,k(1-\lambda(1-k))]^{2n} \qquad \|Sx_0 - p\|}{[1-\lambda(1-L)-L\lambda^2(1-L)]^n \|Sa_0 - p\|} \text{ as} \quad n \to \infty$$

Hence $\{Sx_n\}$converges to $p$faster than $\{S\hat{t}_n\}$ and$\{Sa_n\}$.■

The following theorem shows that Jungck-modified SP-iterative (1.10)converges faster than Jungck-Noor (1.5) and Jungck-Karahan (1.8) to $p \in F$.

3.2 Theorem: Let C be a nonempty closed convex subset of a normed space $X,S.T: C \to C$ be self mappings satisfied Jungck-contraction mappings,assume that $T(C) \subseteq S(C)$.If there exist $\lambda;\ 0 \leq \lambda \leq \beta_n.\alpha_n \leq 1$ for all $n \in \mathbb{N}$, then theJungck-Modified SP-iteration (1.10) converges faster than Jungck-Noor(1.5) and Jungck-Karahan (1.8) to $p \in F$.

Proof: Let $p \in F$, from Jungck-Karahan iterative (1.8), we get

$$\begin{aligned}
\|S\widetilde{w}_{n+1} - p\| &= \|(1-\alpha_n)T\widetilde{w}_n + \alpha_n T\widetilde{u}_n - p\| \\
&\leq L(1-\alpha_n)\|S\widetilde{w}_n - p\| + \alpha_n L\|S\widetilde{u}_n - p\| \\
&\leq L[(1-\alpha_n)\|S\widetilde{w}_n - p\| \\
&\quad + \alpha_n(L(1-\beta_n)\|S\widetilde{w}_n - p\| + L\beta_n(1-\gamma_n)\|S\widetilde{w}_n - p\| + L^2\beta_n\gamma_n\|S\widetilde{w}_n - p\|)] \\
&\leq L\big(1-\alpha_n(1-L) - L\alpha_n\beta_n\gamma_n(1-L)\big)\|S\widetilde{w}_n - p\| \\
&\leq L\big(1-\lambda(1-L) - L\lambda^3(1-L)\big)\|S\widetilde{w}_n - p\| \\
&\ \vdots \\
&\leq [L(1-\lambda(1-L) - L\lambda^3(1-L))]^n\|S\widetilde{w}_0 - p\|
\end{aligned}$$

Let,

$$JKA_n := [L(1-\lambda(1-L) - L\lambda^3(1-L))]^n\|S\widetilde{w}_0 - p\|$$

From Jungck-Noor iterative (1.5), we obtain that

$$
\begin{aligned}
\|Sw_{n+1} - p\| &= \|(1-\alpha_n)Sw_n + \alpha_n Tu_n - p\| \\
&\le (1-\alpha_n)\|Sw_n - p\| + L\alpha_n\|Su_n - p\| \\
&\le (1-\alpha_n)\|Sw - p\| + L\alpha_n(1-\beta_n)\|Sw_n - p\| + L^2\alpha_n\beta_n\|Sv_n - p\| \\
&= (1-\alpha_n)\|Sw_n - p\| + L\alpha_n(1-\beta_n)\|Sw_n - p\| + L^2\alpha_n\beta_n\|(1-\gamma_n)Sw_n + \gamma_n Tw_n - p\| \\
&\le [1-\lambda(1-L) - L\lambda^2(1-L) - L^2\lambda^3]\|Sw_n - p\| \\
&\vdots \\
&\le [(1-\lambda(1-L) - L\lambda^2(1-L) - L^2\lambda^3(1-L))]^n\|Sw_0 - p\|
\end{aligned}
$$

$$
\mathrm{JNO_n} \coloneqq [1-\lambda(1-L) - L\lambda^2(1-L) - L^2\lambda^3(1-L))]^n\|Sw_0 - p\|
$$

From Jungck-modified SP-iteration we have

$$
JMS_n \coloneqq [L(1-\lambda(1-L))]^{2n}\|Sx_0 - p\|
$$

Now since:

$$
\frac{JMS_n}{JKA_n} = \frac{[L(1-\lambda(1-L))]^{2n}\|Sx_0 - p\|}{[L(1-\lambda(1-L) - L\lambda^3(1-L)]^n\|S\widetilde{w}_0 - p\|} \text{as} \quad n \to \infty
$$

$$
\frac{JMS_n}{\mathrm{JNO_n}} = \frac{[k(1-\lambda(1-L))]^{2n} \qquad \|Sx_0 - p\|}{[1-\lambda(1-L) - L\lambda^2(1-L) - L^2\lambda^3(1-L)]^n\|Sw_0 - p\|} \text{as} \quad n \to \infty
$$

Hence $\{Sx_n\}$ converges to $p$faster than $\{S\widetilde{w}_n\}$ and $\{Sw_n\}$.■

The next theorem prove that Jungck-modified SP-iterative (1.10) convergence faster than Jungck-CR (1.7}) and Jungck-SP (1.9) to $p \in F$.

3.3 Theorem: Let C be a nonempty closed convex subset of a normed space X,$S.T: C \to C$ be self mappings satisfied Jungck-contraction mappings,assume that $T(C) \subseteq S(C)$ .If there exist λ ; $0 \le \lambda \le \beta_n.\alpha_n \le 1$ for all $n \in \mathbb{N}$, then the Jungck-Modified SP-iteration (1.10) converges faster than Jungck-CR (1.7) and SP-iteration(1.6)to p ∈ F.

Proof: Let p ∈ F from JCR (1.7) iterative, we obtain that

$$
\begin{aligned}
\|S\widehat{w}_{n+1} - p\| &= \|(1-\alpha_n)S\widehat{u}_n + \alpha_n T\widehat{u}_n - p\| \\
&\le (1-\alpha_n)\|S\widehat{u}_n - p\| + \alpha_n L\|S\widehat{u}_n - p\| \\
&\le (1-\alpha_n(1-L))\|S\widehat{u}_n - p\| \\
&= L(\alpha_n(1-L)\,[L(1-\beta_n)\|S\widehat{w}_n - p\| + +L\beta_n(\|(1-\gamma_n)S\widehat{w}_n + \gamma_n T\widehat{w}_n - p)\|] \\
&\le L(1-\alpha_n(1-L))(1-L\beta_n\gamma_n(1-L))\|S\widehat{w}_n - p\| \\
&\le L(1-\lambda(1-L))(1-\lambda^2(1-L))\|S\widehat{w}_n - p\| \\
&\vdots \\
&\le [L(1-\lambda(1-L))(1-\lambda^2(1-L))]^n\|S\widehat{w}_0 - p\|
\end{aligned}
$$

Let,

$$JCR_{\mathrm{n}} \coloneqq [\, L(1-\lambda(1-L))(1-\lambda^2(1-L))]^n \|S\widehat{w}_0 - p\|$$

From Jungck-SP-iteration (1.9), we obtain that

$$\begin{aligned}
\|S\hat{a}_{n+1} - p\| &= \left\|(1-\alpha_n)S\hat{b}_n + \alpha_n T\hat{b}_n - p\right\| \\
&\leq \left(1-\alpha_n(1-L)\right)\left\|S\hat{b}_n - p\right\| \\
&= (1-\alpha_n(1-L)\,[(1-\beta_n)\|S\hat{c}_n - p\| + +L\beta_n\|Sc_n - p\| \\
&\leq (1-\alpha_n(1-L)\left(1-\beta_n(1-L)\right)\|S\hat{c}_n - p\| \\
&\leq (1-\alpha_n(1-L)\left(1-\beta_n(1-L)\right)\left(1-\gamma_n(1-L)\right)\|S\hat{a}_n - p\| \\
&\leq ((1-\lambda(1-L))^3\|S\hat{a}_n - p\| \\
&\vdots \\
&\leq [(1-\lambda(1-L)]^{3n}\|S\hat{a}_0 - p\|
\end{aligned}$$

Put,

$$JSP_n \coloneqq [(1-\lambda(1-L)]^{3n}\|S\hat{a}_0 - p\|$$

Finally, we get from Jungck-modified SP-iteration

$$JMS_n \coloneqq [\, L\left(1-\lambda(1-L)\right)]^{2n}\|Sx_0 - p\|$$

Now since:

$$\frac{JMS_n}{JSP_n} = \frac{[\, L\left(1-\lambda(1-L)\right)]^{2n}\,\|Sx_0-p\|}{[(1-\lambda(1-L)]^{3n}\|S\hat{a}_0-p\|} \text{as} \quad n \to \infty$$

$$\frac{JMS_n}{JCR_n} = \frac{\left[k\left(1-\lambda(1-L)\right)\right]^{2n}\|Sx_0-p\|}{[\, L(1-\lambda(1-L)(1-\lambda^2(1-L))]^n\|S\widehat{w}_0-p\|} \text{as} \quad n \to \infty$$

Hence$\{Sx_n\}$ converges to p faster than $\{S\widehat{w}_0\}$ and $\{S\hat{a}_0\}$.■

3.4 Theorem: Let Cbe a nonempty closed convex subset of a normed space X, $S.T\colon C \to C$ be self mappings satisfied Jungck-contraction mappings,assume that $T(C) \subseteq S(C)$ .If there exist λ ; $0 \leq \lambda \leq \beta_n.\alpha_n \leq 1$ for all $n \in \mathbb{N}$, then the Jungck-Picard-S (1.9) converge faster than Jungck-modifiedSP-iteration (1.10) to p $\in$ F .

Proof: Let$p \in F$, for Jungck modified SP-iteration (1.10), we obtain

$$\begin{aligned}
\|S\hat{x}_{n+1} - p\| &= \|T\hat{y}_n - p\| \\
&\leq L\|S\hat{y}_n - p\| \\
&= L[(1-\alpha_{n)}\|T\hat{x}_n - p\| + \alpha_n\|T\hat{z}_n - p\|] \\
&\leq L^2[(1-\alpha_{n)}\|T\hat{x}_n - p\| + \alpha_n\|S\hat{z}_n - p\|]
\end{aligned}$$

$$= L^2[(1-\alpha_{n)}\|S\hat{x}_n - p\| + \alpha_n\|(1-\ \beta_n\ )\ S\hat{x}_n +\ \beta_n T\hat{x}_n\ \ - p\|]$$
$$\leq L^2[(1-\alpha_{n)}\ \|S\hat{x}_n - p\| + \alpha_n(1-\ \beta_n)\|S\hat{x}_n - p\| + L\alpha_n\beta_n\|S\hat{x}_n - p\|]$$
$$= L^2\big[\big((1-\alpha_n) + \alpha_n(1-\ \beta_n) + L\alpha_n\beta_n\big)\|S\hat{x}_n - p\|\big]$$
$$\leq\ \ L^2\big(1-\ \lambda^2(1-L)\big)\|S\hat{x}_n - p\|$$
$$\vdots$$
$$\leq\ \ [L^2(1-\ \lambda^2(1-L))]^n\|S\hat{x}_0 - p\|$$

Put,

$$JPS_n :=\ [L^2(1-\ \lambda^2(1-L))]^n\|S\hat{x}_0 - p\|$$

Thus we get fromJungck-modified SP-iteration

$$JMS_n \coloneqq\ [\ L\big(1-\ \lambda(1-L)\big)]^{2n}\|Sx_0 - p\|$$

Now since:

$$\frac{JPS_n}{JMS_n} = \frac{[L^2(1-\lambda^2\ (1-L)]^n\ \|S\hat{x}_0-p\|}{[\ L\big(1-\lambda(1-L)\big)]^{2n}\ \|Sx_0-p\|}\text{as}\quad n\to\infty$$

Hence $\{S\hat{x}_n\}$converges to p faster than$\{Sx_n\}$.■

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